A scalable unstructured 3-D finite volume code for the shallow water equations

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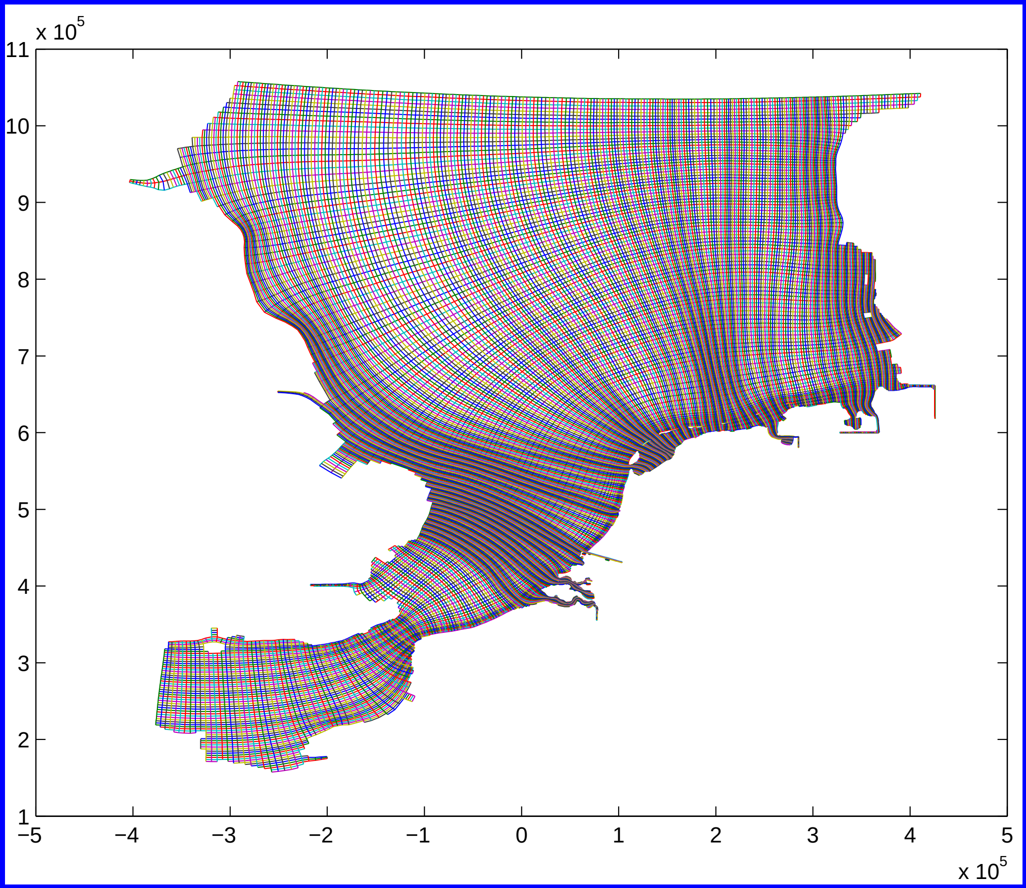
Current Approach

- curvilinear grid.
- 82620 points.
- almost constant resolution.
- stepping on the coast.



Data courtesy of the National Institute for Coastal and Marine Environments (RIKZ).

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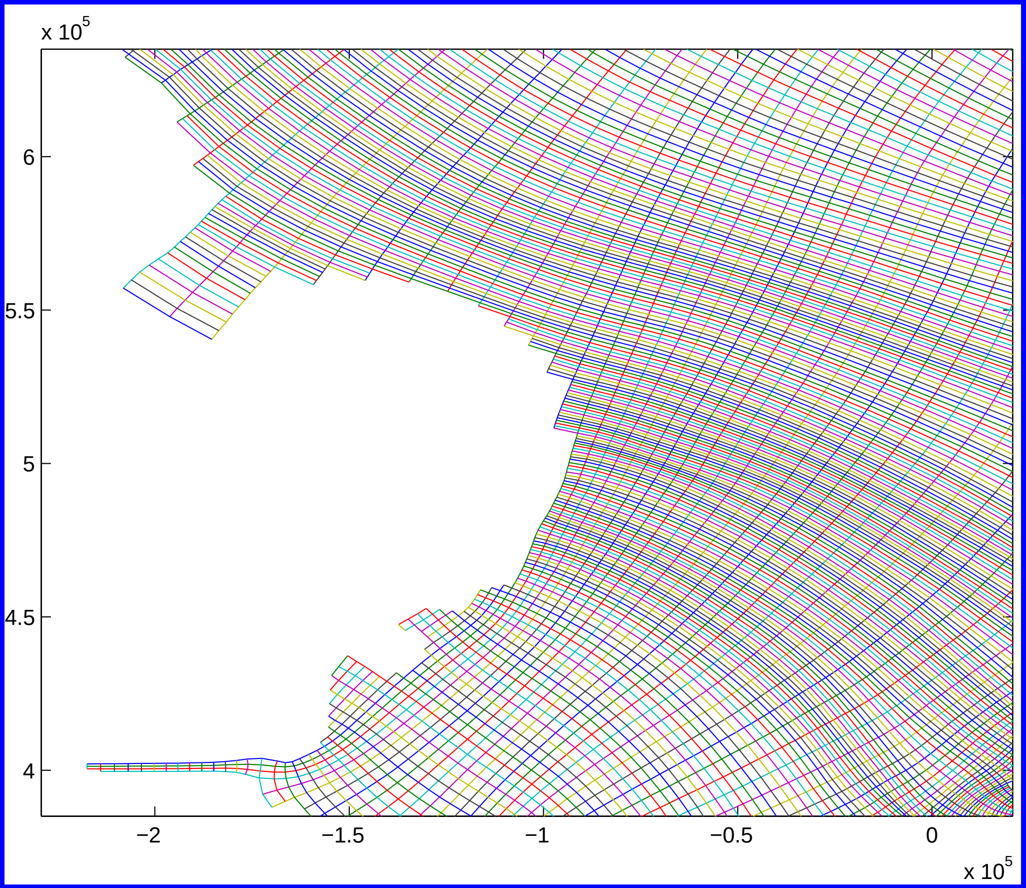

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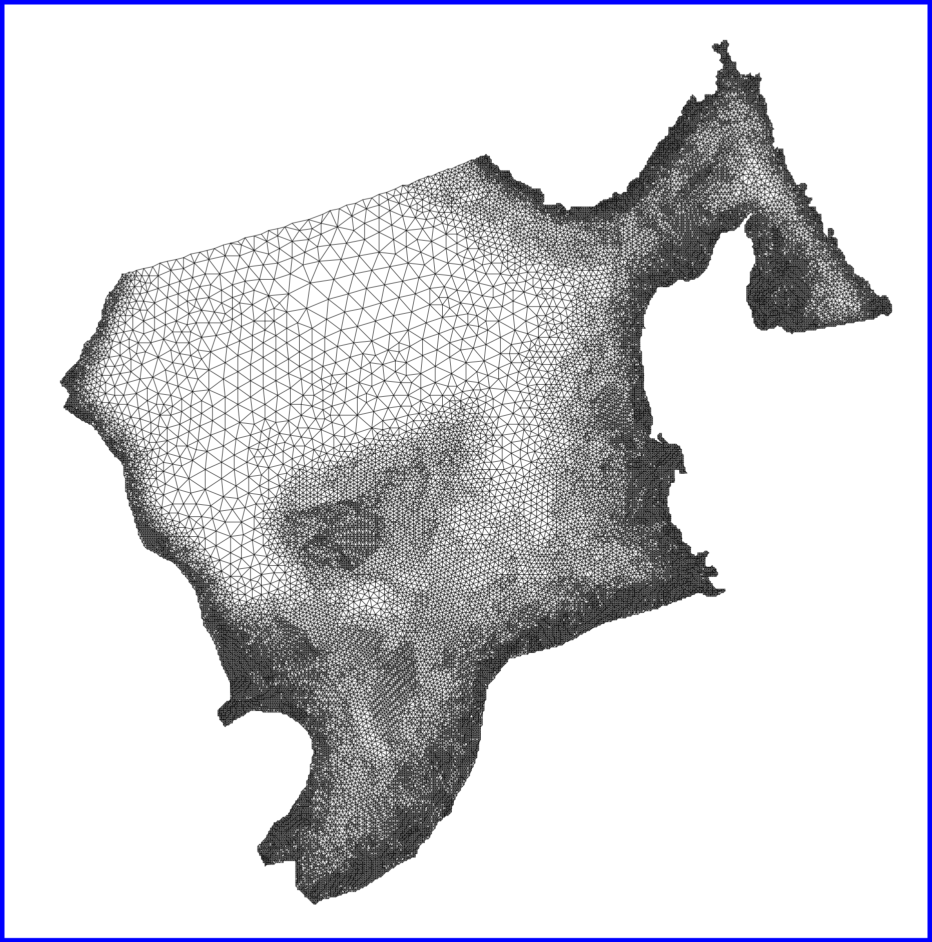
Unstructured grid

- 42050 points.
- highly configurable resolution.
- grid follows coastline.



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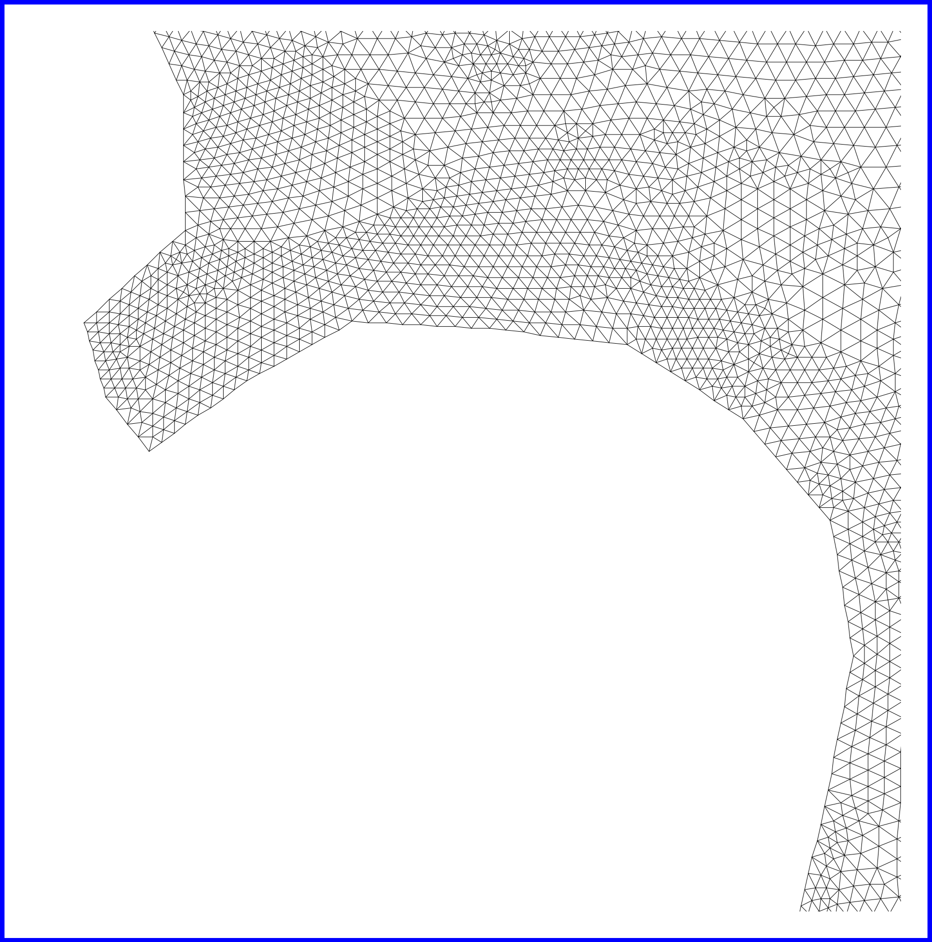
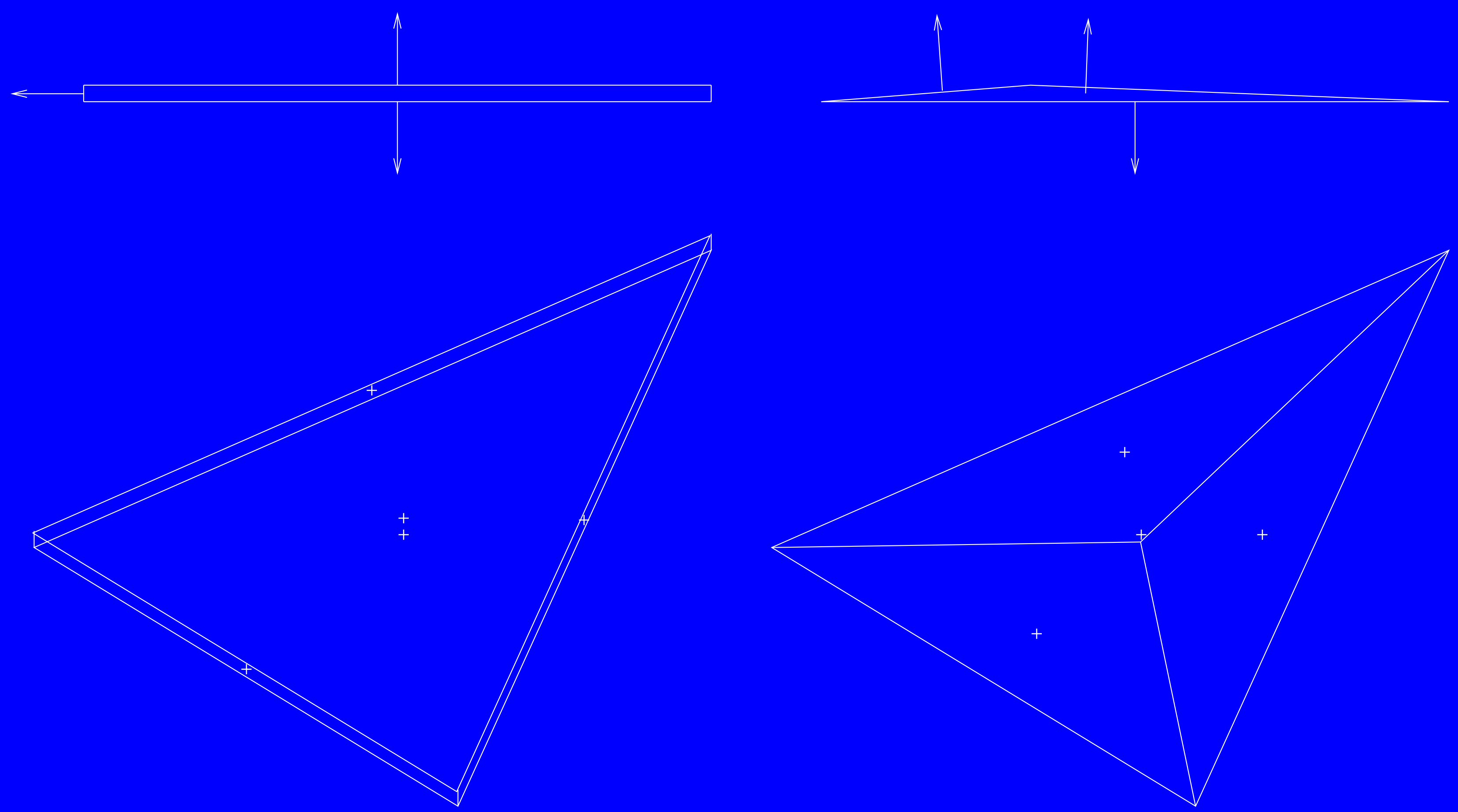


Element shape



The shallow water equations

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\mathrm{D}u_{\mathbf{d}}}{\mathrm{D}t} = \frac{\partial}{\partial \mathbf{d}} \int_{z'=z}^{z'=\eta} g dz' + \nu^h \left(\frac{\partial^2 u_{\mathbf{d}}}{\partial x^2} + \frac{\partial^2 u_{\mathbf{d}}}{\partial y^2} \right) + \frac{\partial}{\partial z} \nu^v \frac{\partial u_{\mathbf{d}}}{\partial z} + (2\mathbf{\Omega} \times \mathbf{u}) \cdot \mathbf{d}$$

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$$\iiint_{g_i} \nabla \cdot \mathbf{u} \, dV = 0$$

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$$\iiint_{g_i} \nabla \cdot \mathbf{u} \, dV = 0$$

$$\iint_{\Gamma g_i} \mathbf{u} \cdot \mathbf{n} \, dA = 0$$

The continuity equation

$$A_i \frac{\partial \eta_i}{\partial t} + \sum_{j \in S_i} \left((\mathbf{n} \cdot \mathbf{N})_{i,j} \sum_{k \in L_j} A_{j,k} u_{j,k} \right) = 0$$

$$A_i \eta_i^{n+1} = A_i \eta_i^n - \theta \Delta t \sum_{j \in S_i} \left((\mathbf{n} \cdot \mathbf{N})_{i,j} \mathrm{A}_j \cdot \mathrm{U}_j^{n+1} \right)$$
$$- (1-\theta) \Delta t \sum_{j \in S_i} \left((\mathbf{n} \cdot \mathbf{N})_{i,j} \mathrm{A}_i \cdot \mathrm{U}_j^{n} \right)$$

Momentum Equation

$$u_{j,k}^{n+1} = Fu_{j,k}^{n} - g\Delta t\left(G_j^{n+1} + H_j^{n}\right)$$
$$+ \frac{\Delta t}{dz_{j,k}^{n}}\left(\nu_{j,k+\frac{1}{2}}^{v}\frac{u_{j,k+1}^{n+1} - u_{j,k}^{n+1}}{\Delta z_{j,k+\frac{1}{2}}^{n}} - \nu_{j,k-\frac{1}{2}}^{v}\frac{u_{j,k}^{n+1} - u_{j,k-1}^{n+1}}{\Delta z_{j,k-\frac{1}{2}}^{n}}\right)$$

$$\mathbf{M}_j^n \mathrm{U}_j^{n+1} = B_j^n - \theta g \Delta t G_j^{n+1} \Delta \mathrm{Z}_j$$

Pressure Discretisation

G_j and H_j are linear functions on the set of surface heights such that $G_j + H_j$ is an approximation to $\left.\frac{\partial p}{\partial \mathbf{n}_j}\right|_j$.

For stability we require that, for a constant pressure field (ie constant surface height), $G_j = H_j = 0$.

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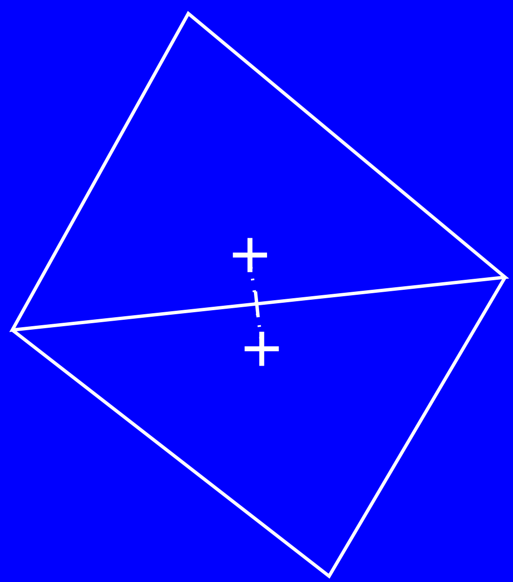
To obtain a symmetric positive definite free surface matrix, we may require $G_j = \alpha \left(\eta_{i(j,1)} - \eta_{i(j,2)}\right)$.

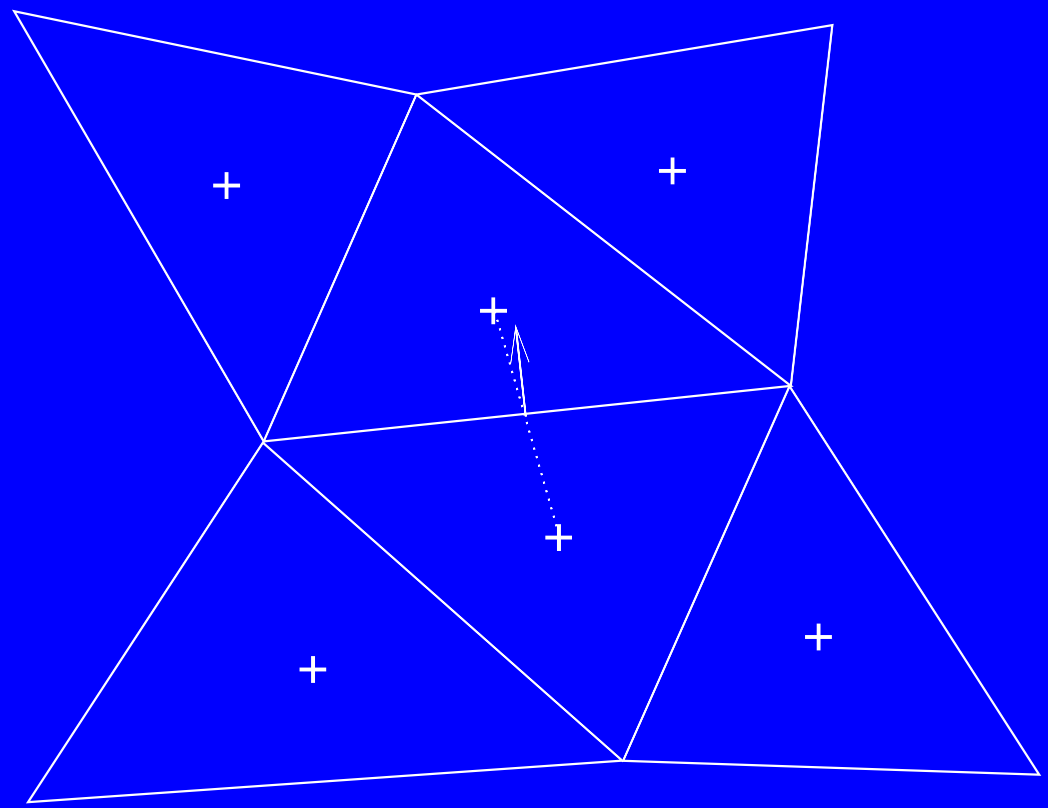
Orthogonal grid



$$G_j = \frac{\eta_{i(j,1)} - \eta_{i(j,2)}}{\left\|\mathbf{x}_{i(j,1)} - \mathbf{x}_{i,(j,2)}\right\|}$$

$$H_j = 0$$

Non-orthogonal grid



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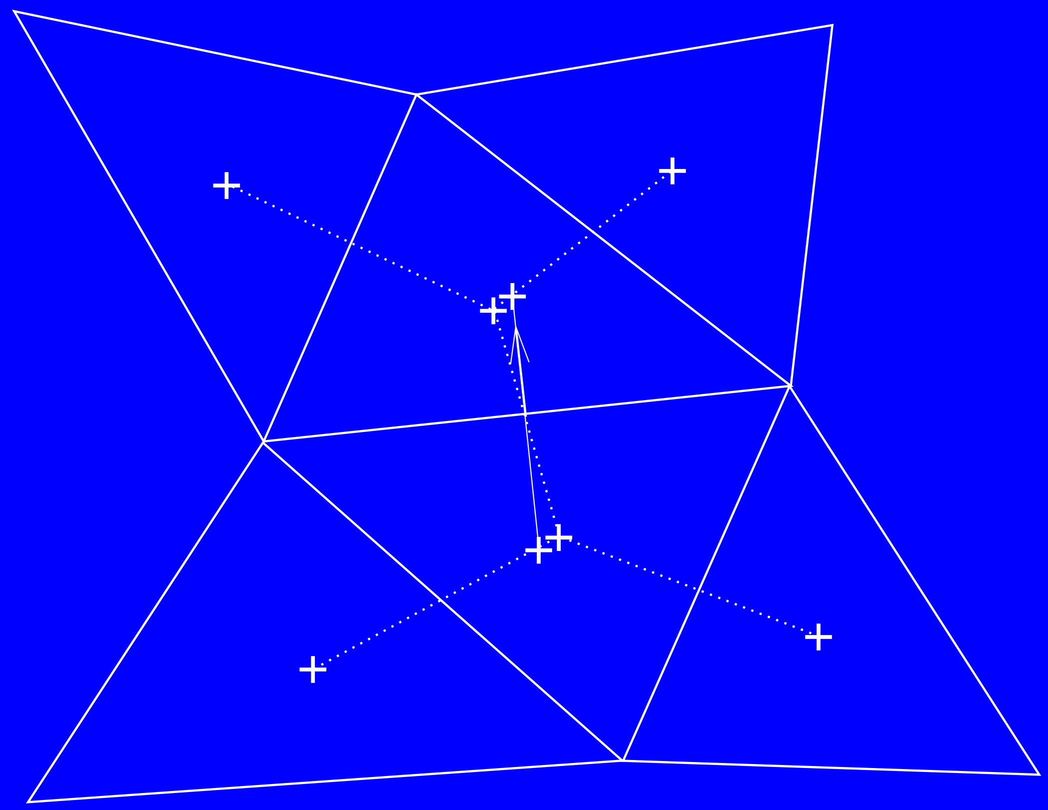
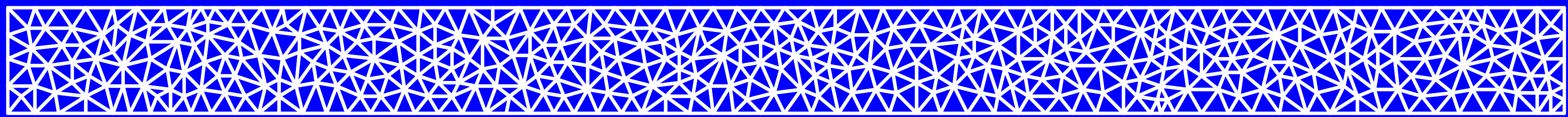


Auxilliary point method.

Hydrostatic jump in a channel



Streamline tracking

Velocity field requirements

1. It must agree with known data.
2. It must be easy to integrate analytically.
3. The streamlines produced by integrating the field must not cross each other or closed boundaries.
4. It should satisfy the continuity equation everywhere.

A Linear Field

$$\mathbf{u}(\mathbf{x}) := \begin{bmatrix} a_x x + b_x \\ a_y y + b_y \end{bmatrix}$$

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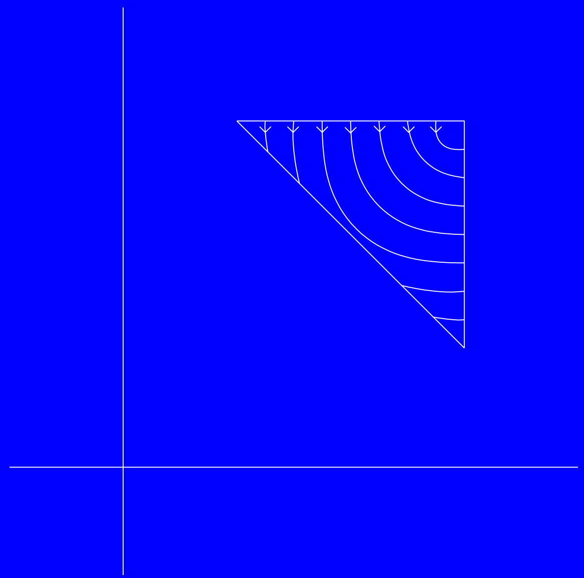
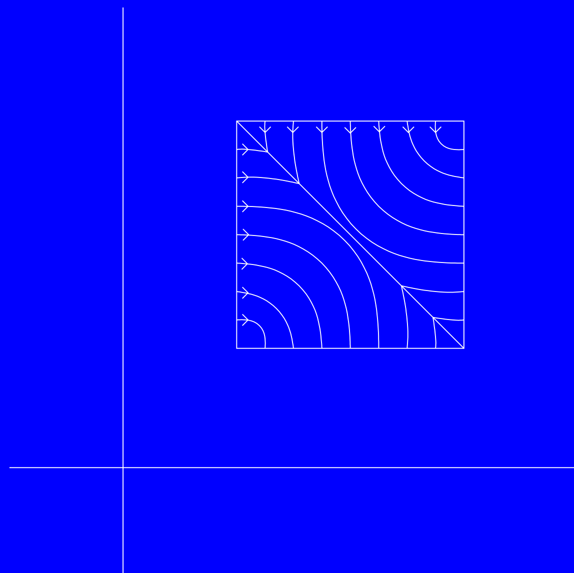
Streamlines leaving and re-entering

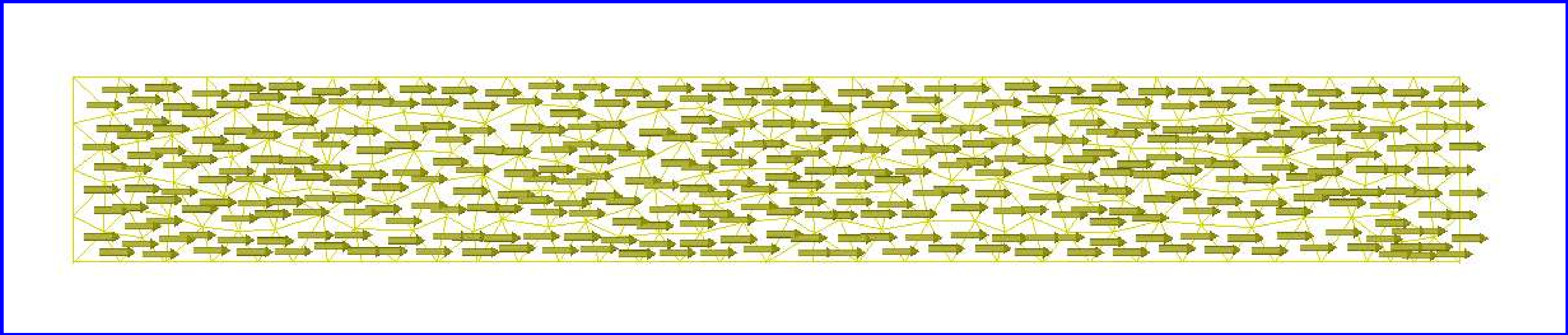


Calculated streamlines for cell face velocities $u(2,3)\mathbf{n} = (0,-2)$, $u(3,2)\mathbf{n} = (2,0)$ and $u(2,2)\mathbf{n} = (0,0)$. The pseudo-velocity field is given by $\mathbf{u}(\mathbf{x}) = (x,-y)$.

Streamlines crossing



Demonstration - channel flow



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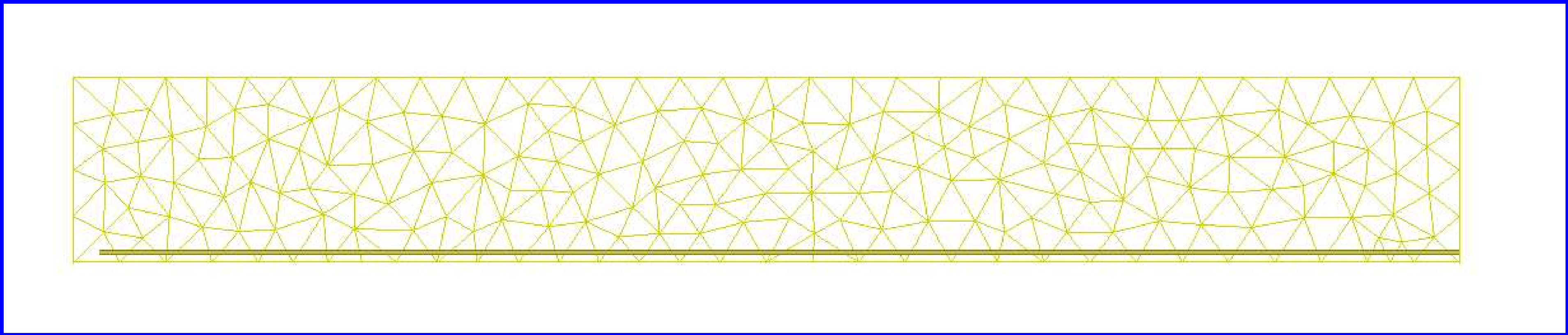
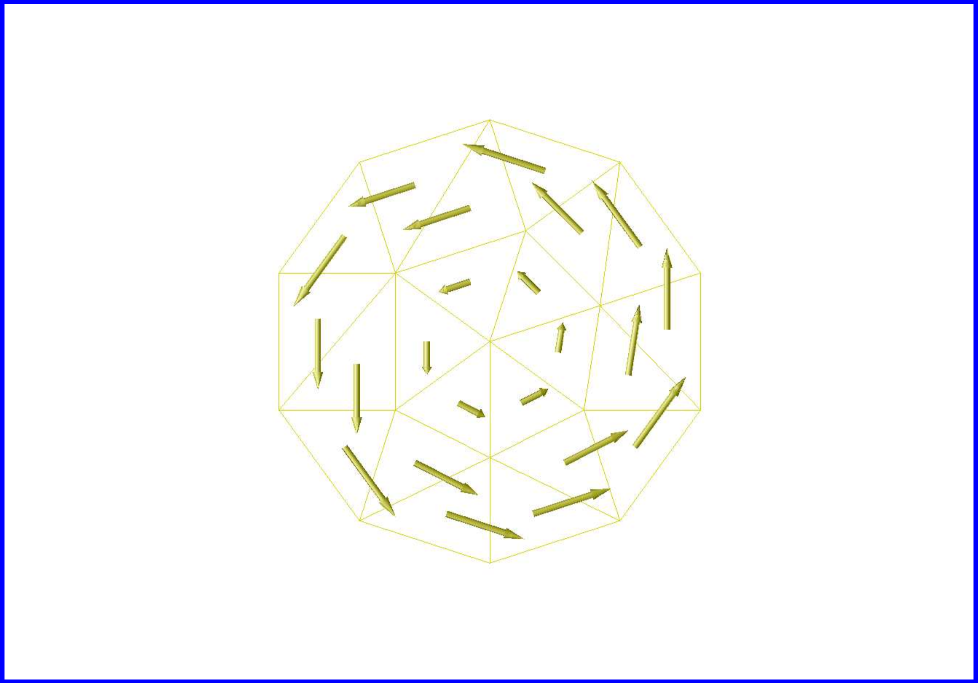


Demonstration - closed loops



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