Deviatoric Stress Tensor

The horizontal components of the divergence of the stress tensor (Wajsowicz, 1993) in nondimesional, orthogonal curvilinear coordinates (ξ, η, s) with dimensional, spatially-varying metric factors ($\frac{1}{m}$, $\frac{1}{n}$, H_z) and velocity components (u, v, ωH_z) are given by

$$\begin{aligned}F^u \equiv \widehat{\xi}\cdot(\nabla\cdot\sigma) = \frac{mn}{H_z}\Bigg[&\frac{\partial}{\partial\xi}\left(\frac{H_z\sigma_{\xi\xi}}{n}\right) + \frac{\partial}{\partial\eta}\left(\frac{H_z\sigma_{\xi\eta}}{m}\right) + \frac{\partial}{\partial s}\left(\frac{\sigma_{\xi s}}{mn}\right) + \\ &H_z\sigma_{\xi\eta}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\right) - H_z\sigma_{\eta\eta}\frac{\partial}{\partial\xi}\left(\frac{1}{n}\right) - \frac{1}{n}\sigma_{ss}\frac{\partial H_z}{\partial\xi}\Bigg]\end{aligned} \tag{1}$$

$$\begin{aligned}F^v \equiv \widehat{\eta}\cdot(\nabla\cdot\sigma) = \frac{mn}{Hz}\Bigg[&\frac{\partial}{\partial\xi}\left(\frac{H_z\sigma_{\eta\xi}}{n}\right) + \frac{\partial}{\partial\eta}\left(\frac{\sigma_{\eta\eta}}{m}\right) + \frac{\partial}{\partial s}\left(\frac{\sigma_{\eta s}}{mn}\right) + \\ &H_z\sigma_{\eta\xi}\frac{\partial}{\partial\xi}\left(\frac{1}{n}\right) - H_z\sigma_{\xi\xi}\frac{\partial}{\partial\eta}\left(\frac{1}{m}\right) - \frac{1}{m}\sigma_{ss}\frac{\partial H_z}{\partial\eta}\Bigg]\end{aligned} \tag{2}$$

where

$$\begin{aligned}\sigma_{\xi\xi} &= (A_M+\nu)\,e_{\xi\xi} + (\nu - A_M)\,e_{\eta\eta},\\ \sigma_{\eta\eta} &= (\nu - A_M)\,e_{\xi\xi} + (A_M+\nu)\,e_{\eta\eta},\\ \sigma_{ss} &= 2\,\nu\,e_{ss},\\ \sigma_{\xi\eta} &= \sigma_{\eta\xi} = 2\,A_M\,e_{\xi\eta},\\ \sigma_{\xi s} &= 2\,K_M\,e_{\xi s},\\ \sigma_{\eta s} &= 2\,K_M\,e_{\eta s},\end{aligned} \tag{3}$$

and the strain field is

$$\begin{aligned}e_{\xi\xi} &= m\frac{\partial u}{\partial\xi} + mnv\frac{\partial}{\partial\eta}\left(\frac{1}{m}\right),\\ e_{\eta\eta} &= n\,\frac{\partial v}{\partial\eta} + mnu\frac{\partial}{\partial\xi}\left(\frac{1}{n}\right),\\ e_{ss} &= \frac{1}{H_z}\frac{\partial(\omega H_z)}{\partial s} + \frac{m}{H_z}u\frac{\partial H_z}{\partial\xi} + \frac{n}{H_z}v\frac{\partial H_z}{\partial\eta},\\ 2\,e_{\xi\eta} &= \frac{m}{n}\frac{\partial(nv)}{\partial\xi} + \frac{n}{m}\frac{\partial(mu)}{\partial\eta},\\ 2\,e_{\xi s} &= \frac{1}{mH_z}\frac{\partial(mu)}{\partial s} + mH_z\frac{\partial\omega}{\partial\xi},\\ 2\,e_{\eta s} &= \frac{1}{nH_z}\,\frac{\partial(nv)}{\partial s}\,+ n\,H_z\frac{\partial\omega}{\partial\eta}.\end{aligned} \tag{4}$$

Here, $A_M(\xi,\eta)$ and $K_M(\xi,\eta,s)$ are the spatially varying horizontal and vertical viscosity coefficients, respectively, and ν is another (very small, often neglected) horizontal viscosity coefficient. Notice that because of the generalized terrain-following vertical coordinates of SCRUM/ROMS, we need to transform the horizontal partial derivatives from constant z to constant s surfaces. And the vertical metric or level thickness is the Jacobian of the transformation, $H_z = \frac{\partial z}{\partial s}$. Also in these models, the *vertical* velocity is computed as $\frac{\omega H_z}{mn}$ and has units of m^3/s.

Transverse Stress Tensor

Assuming transverse isotropy, as in Sadourny and Maynard (1997) and Griffies and Hallberg (2000), the deviatoric stress tensor can be split into vertical and horizontal sub-tensors. The horizontal (or transverse) sub-tensor is symmetric, it has a null trace, and it possesses axial symmetry in the local vertical direction. Then, transverse stress tensor can be derived from (1) and (2) yielding

$$
\begin{aligned}
H_z F^u &= n^2 m \frac{\partial}{\partial \xi}\left(\frac{H_z F^{u\xi}}{n}\right) + m^2 n \frac{\partial}{\partial \eta}\left(\frac{H_z F^{u\eta}}{m}\right) \\
H_z F^v &= n^2 m \frac{\partial}{\partial \xi}\left(\frac{H_z F^{v\xi}}{n}\right) + m^2 n \frac{\partial}{\partial \eta}\left(\frac{H_z F^{v\eta}}{m}\right)
\end{aligned} \tag{5}
$$

where

$$
\begin{aligned}
F^{u\xi} &= \frac{1}{n} A_M \left[\frac{m}{n}\frac{\partial(nu)}{\partial \xi} - \frac{n}{m}\frac{\partial(mv)}{\partial \eta}\right], \\
F^{u\eta} &= \frac{1}{m} A_M \left[\frac{n}{m}\frac{\partial(mu)}{\partial \eta} + \frac{m}{n}\frac{\partial(nv)}{\partial \xi}\right], \\
F^{v\xi} &= \frac{1}{n} A_M \left[\frac{m}{n}\frac{\partial(nv)}{\partial \xi} + \frac{n}{m}\frac{\partial(mu)}{\partial \eta}\right], \\
F^{v\eta} &= \frac{1}{m} A_M \left[\frac{n}{m}\frac{\partial(mv)}{\partial \eta} - \frac{m}{n}\frac{\partial(nu)}{\partial \xi}\right].
\end{aligned} \tag{6}
$$

Notice the flux form of (5) and the symmetry between the $F^{u\xi}$ and $F^{v\eta}$ terms which are defined at density points on a C-grid. Similarly, the $F^{u\eta}$ and $F^{v\xi}$ terms are symmetric and defined at vorticity points. These staggering positions are optimal for the discretization of the tensor; it has no computational modes and satisfy first-moment conservation.

The biharmonic friction operator can be computed by applying twice the tensor operator (5), but with the squared root of the biharmonic viscosity coefficient (Griffies and Hallberg, 2000). For simplicity and momentum balance, the thickness H_z appears only when computing the second harmonic operator as in Griffies and Hallberg (2000).

Rotated Transverse Stress Tensor

In some applications with tall and steep topography, it will be advantageous to reduce substantially the contribution of the stress tensor (5) to the vertical mixing when operating along constant s-surfaces. The transverse stress tensor rotated along geopotentials (constant depth) is, then, given by

$$
\begin{aligned}
H_z R^u &= n^2 m \frac{\partial}{\partial \xi}\left(\frac{H_z R^{u\xi}}{n}\right) + m^2 n \frac{\partial}{\partial \eta}\left(\frac{H_z R^{u\eta}}{m}\right) + \frac{\partial}{\partial s}\left(R^{us}\right) \\
H_z R^v &= n^2 m \frac{\partial}{\partial \xi}\left(\frac{H_z R^{v\xi}}{n}\right) + m^2 n \frac{\partial}{\partial \eta}\left(\frac{H_z R^{v\eta}}{m}\right) + \frac{\partial}{\partial s}\left(R^{vs}\right)
\end{aligned} \tag{7}
$$

where

$$
\begin{aligned}
R^{u\xi} =& \frac{1}{n}\; A_M \left[\frac{1}{n}\; \left(m\frac{\partial(nu)}{\partial\xi} - m\frac{\partial z}{\partial\xi}\frac{1}{H_z}\frac{\partial(nu)}{\partial s}\right) - \frac{1}{m}\left(n\frac{\partial(mv)}{\partial\eta} - n\frac{\partial z}{\partial\eta}\frac{1}{H_z}\frac{\partial(mv)}{\partial s}\right)\right], \\
R^{u\eta} =& \frac{1}{m}A_M \left[\frac{1}{m}\left(n\frac{\partial(mu)}{\partial\eta} - n\frac{\partial z}{\partial\eta}\frac{1}{H_z}\frac{\partial(mu)}{\partial s}\right) + \frac{1}{n}\;\left(m\frac{\partial(nv)}{\partial\xi} - m\frac{\partial z}{\partial\xi}\frac{1}{H_z}\frac{\partial(nv)}{\partial s}\right)\right], \\
R^{us} =& m\frac{\partial z}{\partial\xi}A_M\left[\frac{1}{n}\;\left(m\frac{\partial z}{\partial\xi}\frac{1}{H_z}\frac{\partial(nu)}{\partial s} - m\frac{\partial(nu)}{\partial\xi}\right) - \frac{1}{m}\left(n\frac{\partial z}{\partial\eta}\frac{1}{H_z}\frac{\partial(mv)}{\partial s} - n\frac{\partial(mv)}{\partial\eta}\right)\right] + \\
& n\;\frac{\partial z}{\partial\eta}A_M\left[\frac{1}{m}\left(n\frac{\partial z}{\partial\eta}\frac{1}{H_z}\frac{\partial(mu)}{\partial s} - n\frac{\partial(mu)}{\partial\eta}\right) + \frac{1}{n}\;\left(m\frac{\partial z}{\partial\xi}\frac{1}{H_z}\frac{\partial(nv)}{\partial s} - m\frac{\partial(nv)}{\partial\xi}\right)\right], \\
R^{v\xi} =& \frac{1}{n}\; A_M\left[\frac{1}{n}\;\left(m\frac{\partial(nv)}{\partial\xi} - m\frac{\partial z}{\partial\xi}\frac{1}{H_z}\frac{\partial(nv)}{\partial s}\right) + \frac{1}{m}\left(n\frac{\partial(mu)}{\partial\eta} - n\frac{\partial z}{\partial\eta}\frac{1}{H_z}\frac{\partial(mu)}{\partial s}\right)\right], \\
R^{v\eta} =& \frac{1}{m}A_M\left[\frac{1}{m}\left(n\frac{\partial(mv)}{\partial\eta} - n\frac{\partial z}{\partial\eta}\frac{1}{H_z}\frac{\partial(mv)}{\partial s}\right) - \frac{1}{n}\;\left(m\frac{\partial(nu)}{\partial\xi} - m\frac{\partial z}{\partial\xi}\frac{1}{H_z}\frac{\partial(nu)}{\partial s}\right)\right], \\
R^{vs} =& m\frac{\partial z}{\partial\xi}A_M\left[\frac{1}{n}\;\left(m\frac{\partial z}{\partial\xi}\frac{1}{H_z}\frac{\partial(nv)}{\partial s} - m\frac{\partial(nv)}{\partial\xi}\right) + \frac{1}{m}\left(n\frac{\partial z}{\partial\eta}\frac{1}{H_z}\frac{\partial(mu)}{\partial s} - n\frac{\partial(mu)}{\partial\eta}\right)\right] + \\
& n\;\frac{\partial z}{\partial\eta}A_M\left[\frac{1}{m}\left(n\frac{\partial z}{\partial\eta}\frac{1}{H_z}\frac{\partial(mv)}{\partial s} - n\frac{\partial(mv)}{\partial\eta}\right) - \frac{1}{n}\;\left(m\frac{\partial z}{\partial\xi}\frac{1}{H_z}\frac{\partial(nu)}{\partial s} - m\frac{\partial(nu)}{\partial\xi}\right)\right].
\end{aligned}
\tag{8}
$$

Notice that transverse stress tensor remains invariant under coordinate transformation. The rotated tensor (7) retains the same properties as the unrotated tensor (5). The additional terms that arise from the slopes of s-surfaces along geopotentials are discretized using a modified version of the triad approach of Griffies *et al.* (1998).

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